Vertex-to-vertex Packings of Congruent Triangles

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Vertex-to-vertex Packings of Congruent Triangles

von **Heiko Harborth** and **Meinhard Möller,** Braunschweig*

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1. Introduction

A packing of congruent triangles in the plane will be called complete (in [4] saturated) if every vertexpoint belongs to exactly two of the triangles, and other common points of the triangles do not exist. What is the minimum number $M(\alpha,\beta)$ of congruent triangles with angles $\alpha \leq \beta \leq \gamma$ (or side lengths $a \leq b \leq 1$, respectively) such that a complete packing of those triangles exists? It may be challenging to puzzle out minimal complete packings on a table playing with congruent triangles cutted out of pasteboard.

For 42 equilateral triangles a complete packing is presented in Figure 1.

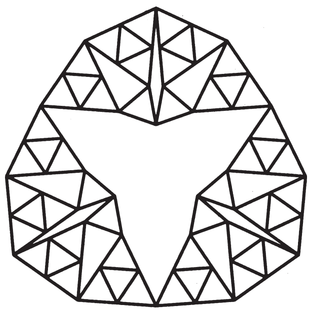

Figure 1

This proves $M(\frac{\pi}{3},\frac{\pi}{3}) \leq 42$ and equality is conjectured (see [1, 2, 3]). From a proof in [4] it follows that $M(\alpha, \beta) \geq 6$ with equality for certain classes of triangles. In general, we can construct complete packings with less then 42 triangles whenever the triangle is not equilateral.

Theorem 1. For any nonequilateral triangle there exists a complete packing that implies $M(\alpha, \beta) \leq 20$.

If the triangle is not an isosceles triangle with $\gamma \geq \frac{5\pi}{6}$ than we obtain an even smaller bound.

Theorem 2. For any triangle, excluded equilateral triangles and isosceles triangles with $\gamma \geq \frac{5\pi}{6}$, there exists a complete packing that implies $M(\alpha,\beta) \leq 18$.

The proofs that the constructed complete packings determine the minimum numbers of triangles are missing and at this moment we do not see a possibility how to manage such proofs.

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2. Proofs of Theorems 1 and 2

We have to consider all pairs (α, β) of the triangular region in Figure 2 where $\alpha = 0$ and $(\alpha, \beta) = (\frac{\pi}{3}, \frac{\pi}{3})$ are excluded. Both

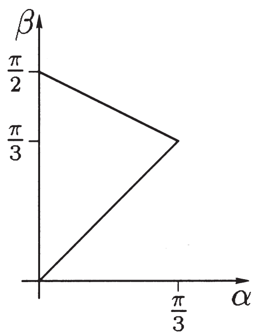

Figure 2

proofs are given by the unions of the following general classes of triangles:

(i) If $\alpha + \beta \leq \frac{\pi}{4}$ and $\alpha \neq \beta$ then $M(\alpha,\beta) \leq 12$.

(ii) If $\alpha + \beta > \frac{\pi}{6}$ then $M(\alpha, \beta) \leq 18$.

(iii) If $\alpha + \beta < \frac{5\pi}{12}$ then $M(\alpha, \beta) \leq 20$.

The following constructions prove these three cases.

(i): Consider three of the congruent triangles composed as in Figure 3. A reflection of these three triangles to the vertical dashed line is possible since $\psi = \pi - \gamma - 2\beta = \frac{\pi}{2} + \alpha - \beta < \frac{\pi}{2}$. If $\alpha + \varphi + \beta < \frac{\pi}{2}$ then a reflection to the horizontal dashed line also is possible and we have obtained a complete packing of twelve triangles. With $\alpha < \beta$ and $\beta < \frac{\pi}{4}$, the condition is fulfilled since

$$\alpha + \varphi + \beta < \alpha + \beta + \frac{\frac{\pi}{2} - \psi}{2} = \frac{\alpha + 3\beta}{2} < 2\beta < \frac{\pi}{2}.$$

An example of 12 triangles is given in Figure 4.

(ii): Consider three triangles composed as in Figure 5. A reflection to the horizontal dashed line is possible since $2\gamma + \frac{\pi}{3} > \pi$ and $2\gamma + \frac{\pi}{3} < 2\pi$. We obtain six triangles with only two free vertexpoints. The dashed line at an angle $\frac{\pi}{3}$ to the horizontal dashed line and through one of the free vertexpoints (see Figure 5) does not intersect the six triangles since $\varphi + \beta = \gamma - \frac{\pi}{3} + \beta = \frac{2\pi}{3} - \alpha > \frac{\pi}{3}$ and since $\gamma > \frac{\pi}{3}$. Thus three copies of the six triangles can be composed to become a complete packing of 18 triangles (see Figure 6 as an example). – It may be remarked that for some triangles even two copies of the six triangles can be composed successfully, for example, if $\gamma > \frac{\pi}{2}$ and $\alpha < \frac{\pi}{6}$.

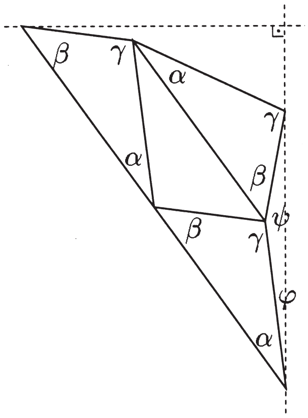

Figure 3

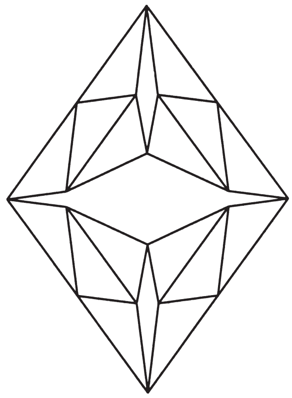

Figure 4

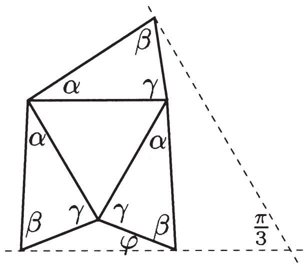

Figure 5

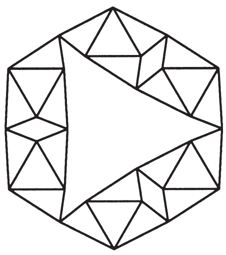

Figure 6

(iii): Four triangles are composed around a rhomb of side length a, the smallest side length of the triangle (see Figure 7). Then the four remaining free vertexpoints of the triangles are the vertexpoints of a parallelogram. Let ε be an angle of this rhomb. Then the four triangles are completely inside of the parallelogram if $\alpha < \varepsilon < \pi - \alpha$ is fulfilled since this implies that $\beta + \gamma + \varepsilon > \pi$ and $\beta + \gamma + \pi - \varepsilon > \pi$. Now ε can be chosen in such a way that the side lengths for one pair of opposite sides of the parallelogram are $2b$, that is, there is a triangle $(1, b, 2b)$ at the vertex of the rhomb with the angle ε. This is possible since on the one hand the triple $(1, b, 2b)$ fulfills the triangle inequalities which follows from $a \le b \le 1$ and $\alpha + \beta < \frac{\pi}{2}$. On the other hand it is $\alpha < \varepsilon < \pi - \alpha$ fulfilled since for $(1, b, 2b)$ with $\frac{1}{2} < b < 1$ it follow

$$-1 < \cos(\pi + \alpha - \varepsilon) = \frac{1 - 3b^2}{2b} < \frac{1}{4}$$

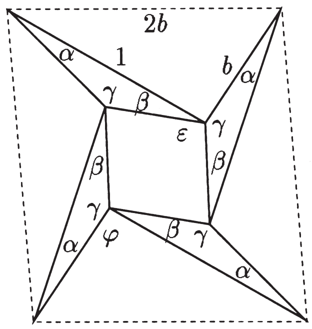

Figure 7

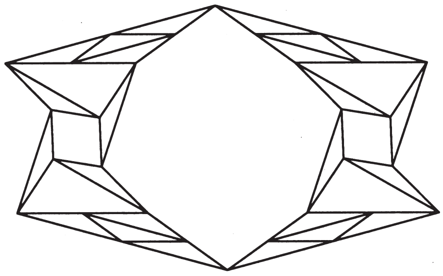

Figure 8

and

$$\frac{5\pi}{12} < \pi + \alpha - \varepsilon < \pi .$$

This implies immediately $\varepsilon > \alpha$, and $\varepsilon < \frac{7\pi}{12} + \alpha \le \pi - \alpha$ follows with $2\alpha \le \alpha + \beta < \frac{5\pi}{12}$.

Now at the two sides of length $2b$, triangles of three copies of the triangles are added to the parallelogram as in Figure 8 to obtain ten triangles.

Since $\alpha + \beta < \frac{\pi}{2}$, that is, $\gamma > \frac{\pi}{2}$, two copies of these sets of ten triangles can be composed as in Figure 8.

3. Special constructions

For several types of triangles there exist special complete packings which imply smaller upper bounds for $M = M(\alpha, \beta)$ than the general bounds of Theorems 1 and 2. Examples with $M = 6$ are discussed in [4] (see Figure 9). For triangles with $\gamma = \frac{2\pi}{3}$ we have $M \le 8$ in Figure 10. For triangles which fit together as in Figure 11 we have $M \le 10$.

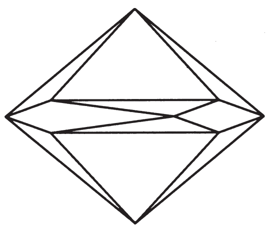

Figure 9

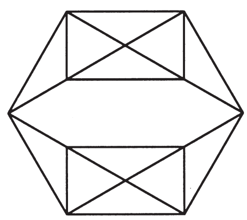

Figure 10

Figures 12 and 13 show two examples with $M \leq 12$. In Figures 14 and 15 there are two examples with $M \leq 14$. Five examples with $M \leq 16$ are presented in Figures 16 to 20. Figure 21 gives an example with $M \leq 18$.

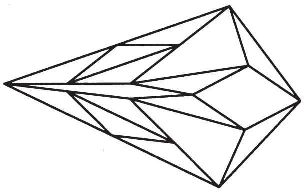

Figure 11

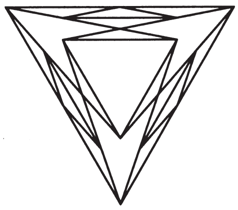

Figure 12

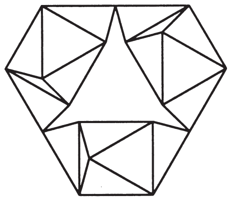

Figure 13

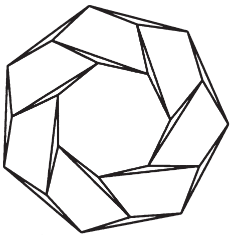

Figure 14

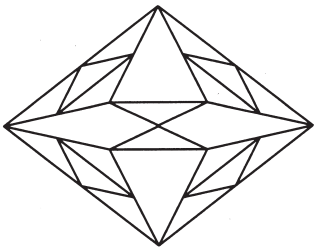

Figure 15

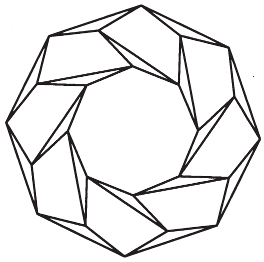

Figure 16

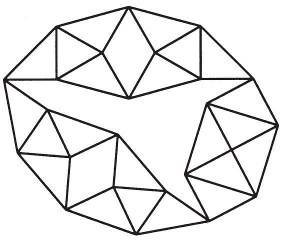

Figure 17

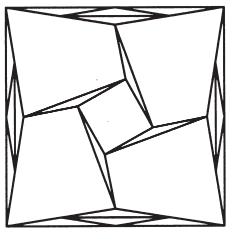

Figure 18

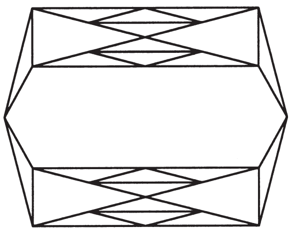

Figure 19

All examples in Figures 10 to 21 are presented only in the cases of isosceles triangles. If $\alpha \neq \beta$ then these examples are also possible so that the corresponding bounds for M are valid for curves in the triangular region of Figure 2.

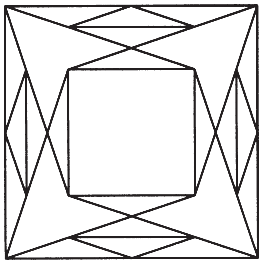

Figure 20

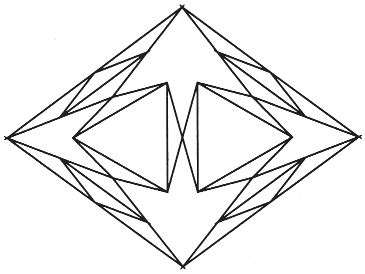

Figure 21

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